Importance Sampling for Errors in an Underdamped Actively Mode-Locked Laser

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Overview

Soliton-based system (Perturbed NLSE with noise)

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Optimal biasing for position-slip errors

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Importance sampled MC simulations with full PDE

How do you optimally exit a potential well with a stable spiral?

Solitons

The nonlinear Schrödinger equation (NLSE) admits the well-known soliton solution

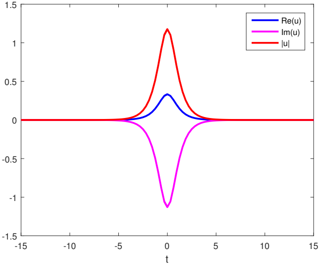
$$u_s(z,t) = E\,\text{sech}[E(t - T(\Omega,z))]\exp[i\Theta(\Omega,T,z,t)].$$

E - amplitude

Ω - frequency or pulse group velocity

T - position

Θ - total phase



Mode-Locked Laser Model

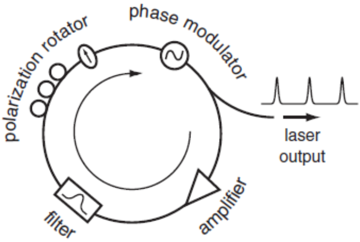
$$\boxed{\text{NLSE}} = a\frac{\partial^2 u}{\partial t^2} - c_1 u + c_2|u|^2 u - c_3|u|^4 u + ib\cos(\omega t)u + \sum_{n=1}^{N_a} f_n(t)\delta(z-n)$$

Amplifiers add zero-mean Gaussian white noise

$$\mathbb{E}[f_i(t)] = 0$$
$$\mathbb{E}[f_i(t)f_j^\dagger(t')] = \sigma^2\delta(t-t')\delta_{ij}$$

- $-c_1 u + c_2|u|^2 u - c_3|u|^4 u$ - linear loss and polarization rotator
- $a\frac{\partial^2 u}{\partial t^2}$ - filtering
- $ib\cos(\omega t)u$ - active phase modulation

Model is underdamped



Schematic courtesy of G.M. Donovan

Damped Oscillator for Soliton Position

To leading order, the perturbations induce evolution equations in the soliton parameters

$$\frac{d\Omega}{dz} = A\,\Omega(z) + B\sin\left(\omega T(z)\right) + \eta_\Omega(z),$$
$$\frac{dT}{dz} = \beta\,\Omega(z) + \eta_T(z), \qquad (\star)$$

where $\eta_\Omega(z)$ and $\eta_T(z)$ are contributions from the noise.

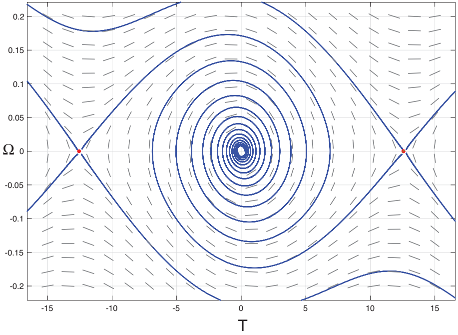
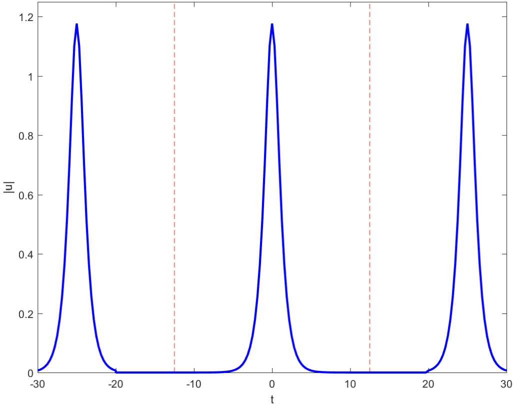


Figure: Phase plane for $(\star)$ without noise, with a stable spiral at the bit-slot center and saddles at bit-slot boundaries.

Position Slip Error Mode

Here we will consider errors where noise causes the pulse to move out of its bit slot. The goal is to compute the probability of this happening.



Importance Sampled Monte Carlo Simulations

Importance sampling allows for the calculation of low probabilities in Monte Carlo simulations by introducing a biasing distribution which is more likely to give events of interest.

$$\begin{aligned}\mathbf{P} = \mathbb{E}[g(X)] &= \int_{-\infty}^{\infty} g(x)p(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} g(x)\frac{p(x)}{p^*(x)}p^*(x)\,\mathrm{d}x = \mathbb{E}^*[g(X)\frac{p(X)}{p^*(X)}]\end{aligned}$$

Calculate estimate for probabilities by weighting with likelihood ratio:

$$\hat{\mathbf{P}}^* = \frac{1}{N}\sum_{j=1}^{N} g(X_j^*)\frac{p(X_j^*)}{p^*(X_j^*)}$$

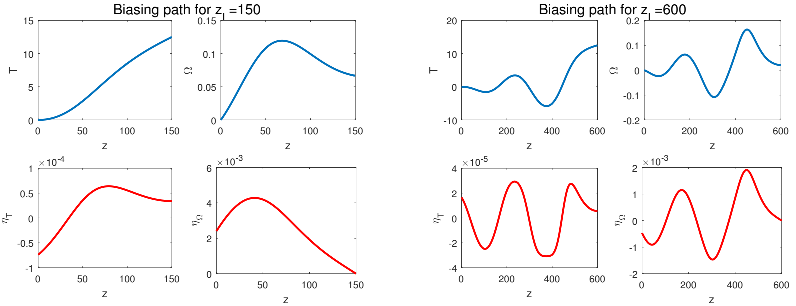
Pick biasing distribution by shifting the mean of Gaussian noise to regions of state space where large position slips are most likely to occur.

Constrained Optimization Problem

In order to find most probable noise configuration leading to a position slip in the system from $\Omega(0) = \Omega_0$, $T(0) = T_0$ to $T(z_L) = \hat{T}$, seek to minimize the action functional:

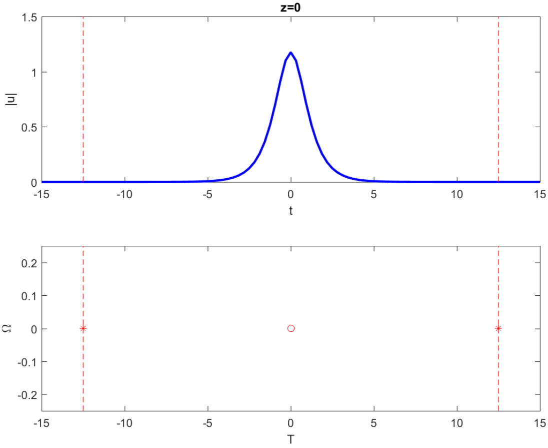
$$S = \frac{1}{2}\int_0^{z_L} C\eta_\Omega(z)^2 + D\eta_T(z)^2 \,\mathrm{d}z$$

subject to the differential constraints from $(\star)$. Using calculus of variations, this problem can be converted into a multipoint BVP that is solved numerically.



Expected Behavior of Biasing Paths without Noise

Biasing paths are direct for short distances and oscillatory for longer distances.



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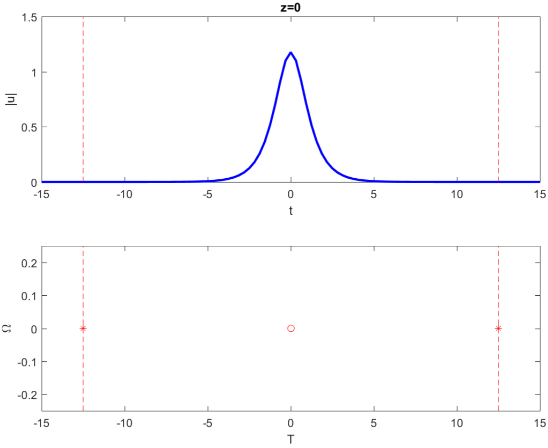
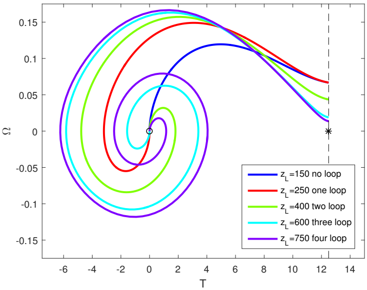
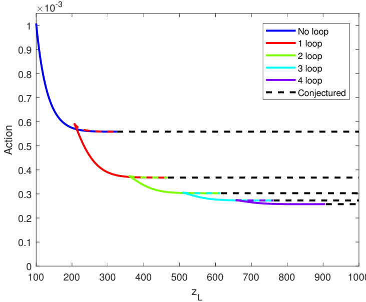


Multiple Biasing Paths Exist

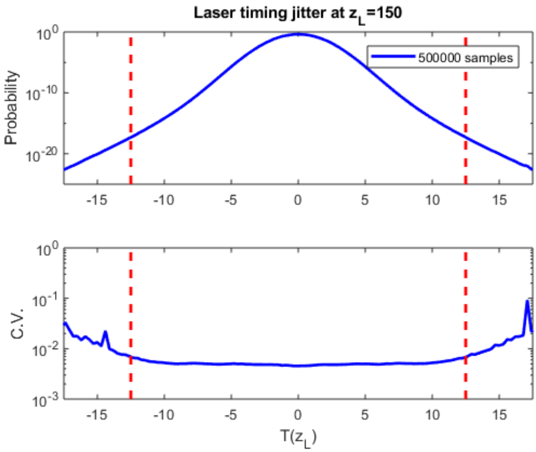
- Paths become increasingly oscillatory as propagation distance increases
- Multiple paths (local minima) coexist at same distance
- Path with most possible oscillations at that distance is typically the global minimum, but other paths with less oscillations still exist and are local minima
- Biasing paths arise through bifurcation from infinity





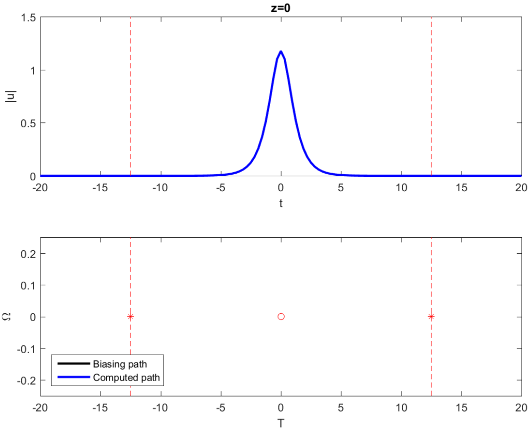
Timing Jitter Results Only Good for Short Propagation Distances

- 10^5 samples $\rightarrow$ 10^{-20} probabilities
- Coefficient of variation (C.V.) indicates simulation convergence
- C.V. shows good convergence at boundary for short distances, worse for longer distances



Deviation from Paths in Simulation

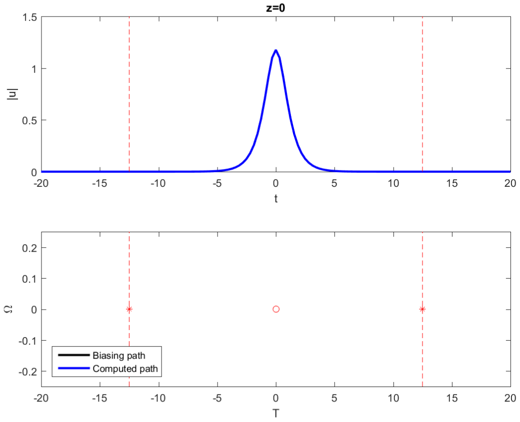
Accumulated deviations from the biasing path can cause the pulse to miss its target by wide margins.



Targeted Importance Sampling

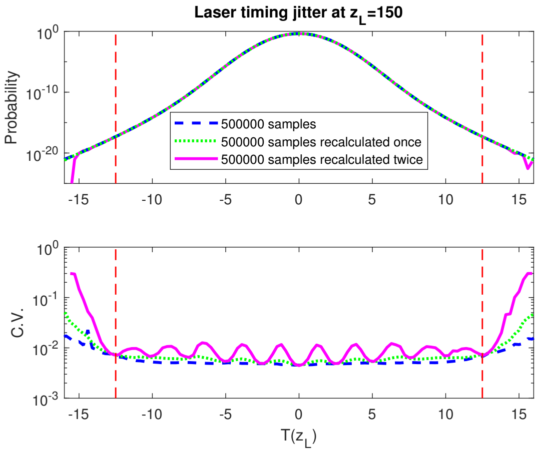
Idea is to recalculate path during simulation. More samples land in desired locations (escape the bit slot) which decreases the sample variance.

- Recalculate path mid-simulation by changing boundary conditions in BVP and re-solving
- Also known as dynamic, adaptive, or closed-loop IS



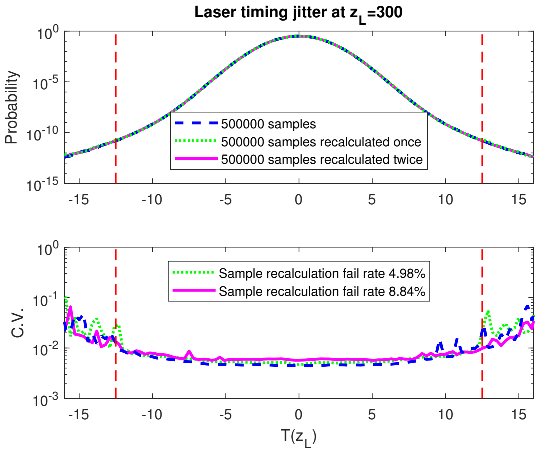
Targeting Results for Short Distance

- Samples tightly cluster near target
- Convergence worse away from target
- No improvement for short distances (convergence already good)



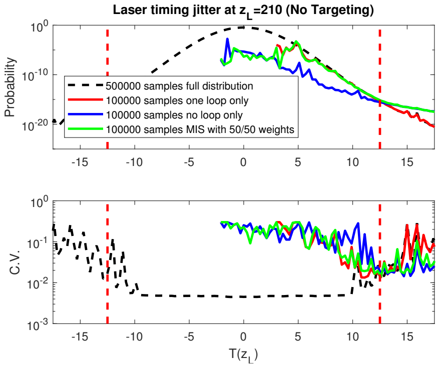
Targeting at Longer Distances

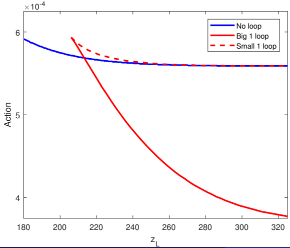
- Recalculating path difficult for longer distances
- Path recalculation fails more the more you target
- Convergence improvement modest with low number of recalculations

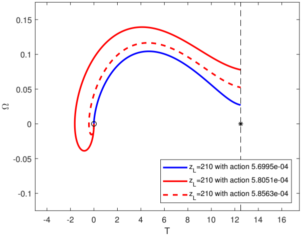


Observation: Multiple Paths Important for IS

- Particular conditions have multiple relevant exit paths
- Can estimate path weights a priori using asymptotic estimates for their likelihood
- Including multiple paths can give correction to probability, and help improve convergence







Precomputed Path Library

- Library of solutions to biasing ODEs created using numerical continuation
- Draw from library when recalculating, rather than re-solving the ODEs
- Encompasses region of phase space with two possible paths
- Shows regions where both path types exist

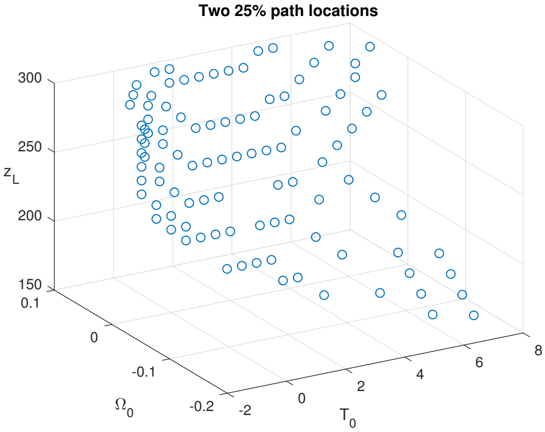
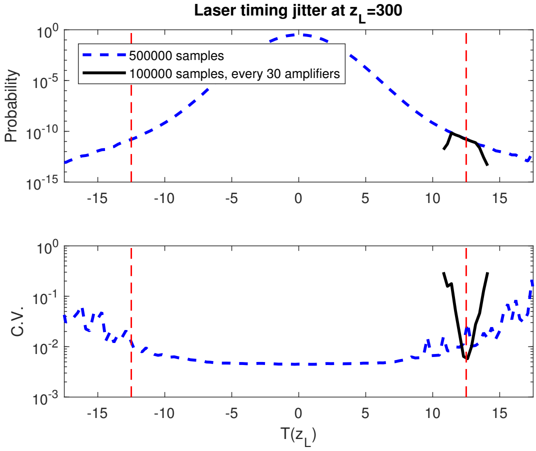


Figure: Locations in (T_0, Ω_0, z_L) where two paths with at least 25% relative probability exist.

Targeted Importance Sampling with Library

- Allows for very tight targeting
- Convergence substantially improved
- Allows for both paths to be taken into account, when relevant
- Only targeting exit region, other targets can be added to resolve the full distribution



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Thanks for your attention!

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